Graph and derivative

25^{th} October 2005

Definition 1. Let n be a positive integer. Then x^n means that x is multiplied to itself n times.

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Definition 2. A *monomial* is an expression cosisting a real-numbered coefficient times one or more variables each raised to the power of a positive integer. Adding and subtracting monomials to one another give us a *polynomial*. A *univariate polynomial* is a polynomial of one variable, of the form $a_n x^n + \ldots + a_1 x + a_0$, and its degree is the highest power of that variable, that is n. The degree of a polynomial is sometimes known as its *order*. A polynomial can be simplified into a product of two polynomials through the process called *factoring*. A *polynomial equation* is an equation of the form $p(\cdot) = 0$, where $p(\cdot)$ is a polynomial.

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Definition 3. A *quadratic equation* is a second-degree polynomial equation, that is one of the form $ax^2 + bx + c = 0$, where a, b and c are constants and $a \neq 0$.

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Example 1. The factors of $mx^2 + nx + p$ are $ax + b$ and $cx + d$, where $ac = m$, $bd = p$ and $ad + bc = n$. Quadratic equations may be solved by factoring.

Example 2. A quadratic polynomial of the form $x^2 + bx$ can be transformed into the form $(x + a)^2$ by adding $(b/2)^2$ and then factor the result. *Completing the square* in the quadratic equation $ax^2 + bx + c = 0$ leads one to

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

which gives us the *quadratic formula*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Definition 4. At a point on the graph where the first derivative is zero, if the second derivative is positive, then that point is a *local minimum*; if the second derivative is negative, it is a *local maximum*; on the other hand, if the second derivative is either zero or is undefined, it is an *inflection point*. An *asymptote* is a straight line to which a non-linear curve smoothly approach as it goes towards infinity, never reaching it. At any point on a graph, if the first derivative is positive the function at that point is *increasing*, and if negative it is *decreasing*. At any point on a graph, if the second derivative is positive it is said that the function is *convex* at that point, and if negative the latter is said to be *concave*.

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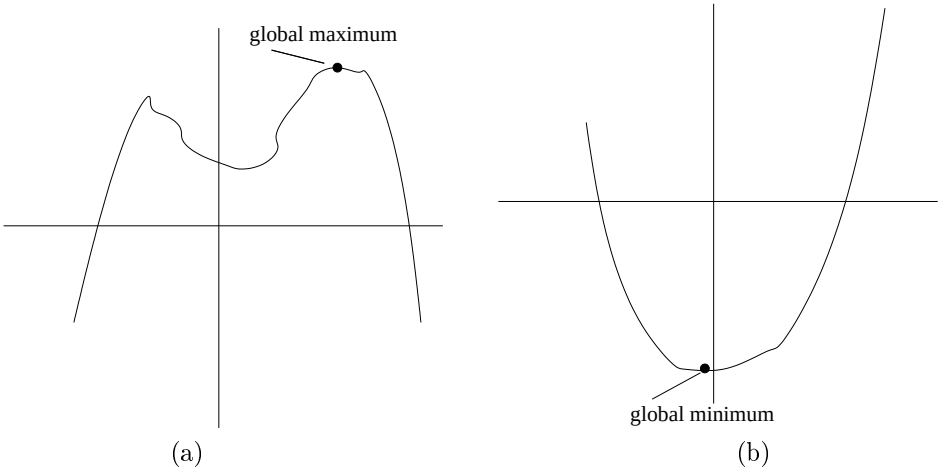


Figure 1 *(a) global maximum-, and (b) global minimum points of a function.*

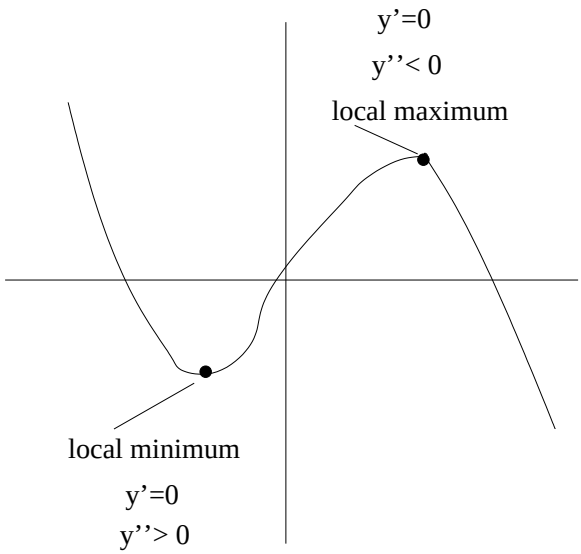
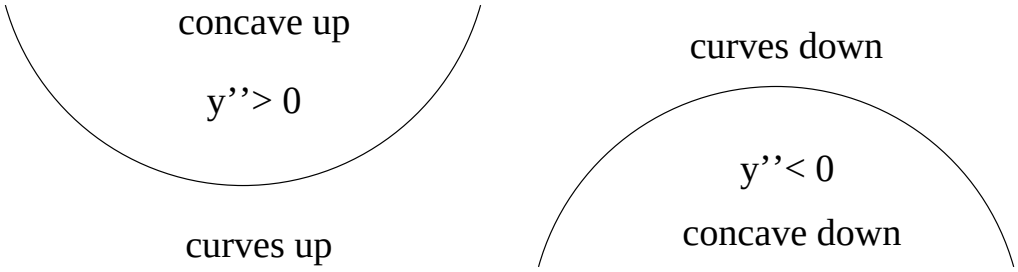
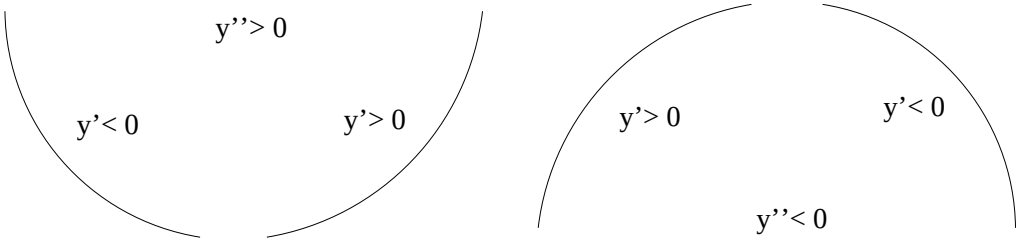


Figure 2 *Local minimum- and local maximum points of a function.*



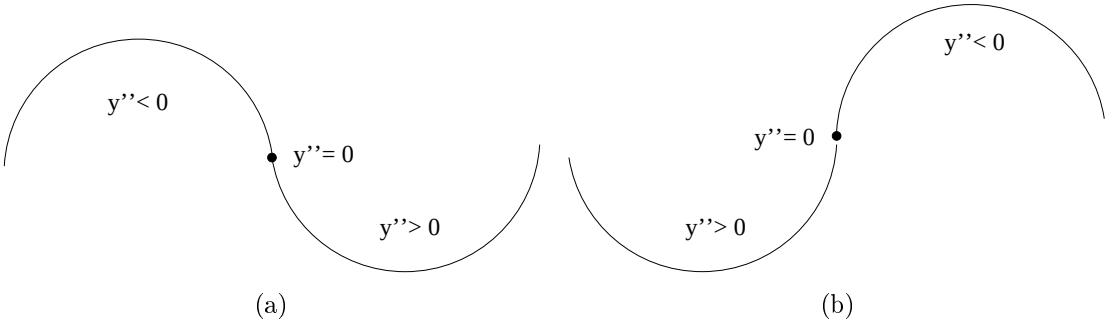
(a) (b)

Figure 3 *The two curvature types, namely (a) concave up ($y'' > 0$) and, (b) concave down ($y'' < 0$).*



(a) (b)

Figure 4 *The four possible curvatures in two dimensions, considering both y' and y'', (a) $y'' > 0$ and, (b) $y'' < 0$.*



(a) (b)

Figure 5 *Inflection points, where $y'' = 0$, (a) with y'' increasing and, (b) with y'' decreasing.*

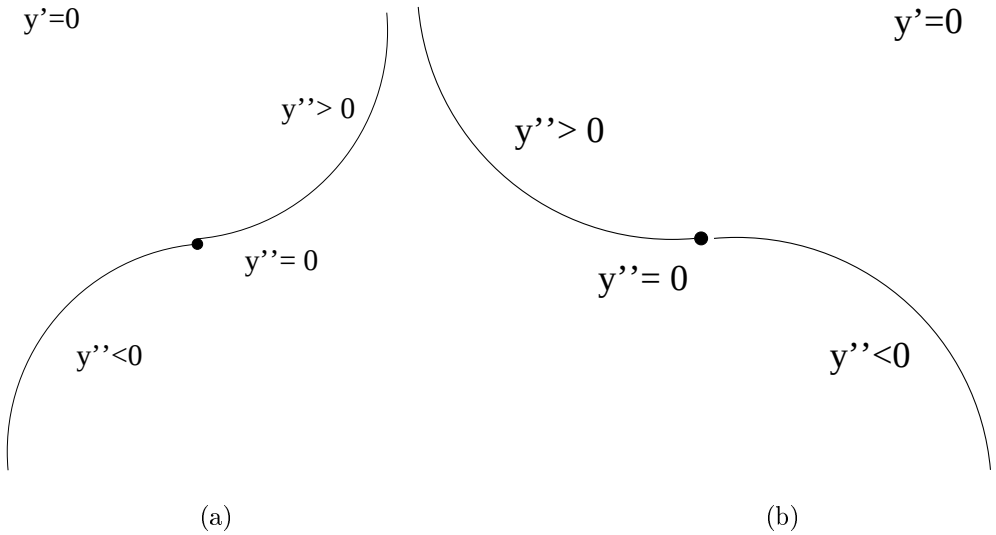


Figure 6 *Stationary inflection points, where both $y' = 0$ and $y'' = 0$, (a) with y'' increasing and, (b) with y'' decreasing.*

Example 3. The cubic function of the form $y = ax^3 + bx^2 + cx + d$ has local maximum and minimum, when these exist, at the point where the first derivative is zero, that is when $3ax^2 + 2bx + c = 0$. At such points,

$$x = \frac{-b \pm \sqrt{b^2 - 3ac}}{3a}$$

Depending on whether $b^2 - 3ac$ is positive or negative, the local minimum and maximum either exist or do not exist. As examples, when $a = 1$, $b = 2$, $c = 3$ and $d = 4$ this value is -5, and therefore the value of x becomes complex. In the space of real number this means that there is neither a maximum- or a minimum point. Instead, in this case we have an inflection point, which is the point where the second derivative becomes zero, that is to say, $6ax + 2b = 0$, or $x = -\frac{b}{3a}$. Figure 7 shows a plot of this case.

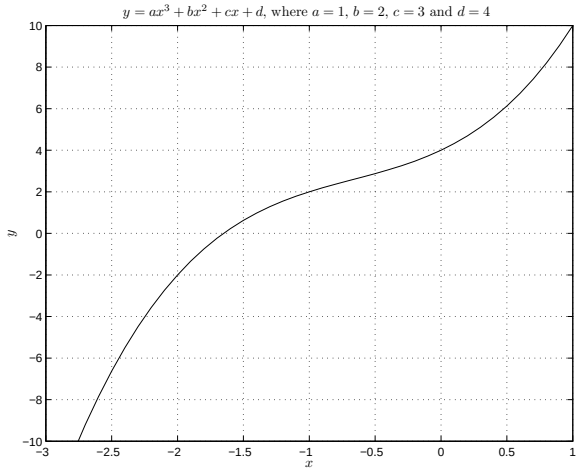


Figure 7 *The cubic function $y = x^3 + 2x^2 + 3x + 4$.*

Figure 8 shows the case where $a = 1$, $b = 5$, $c = 4$ and $d = 3$

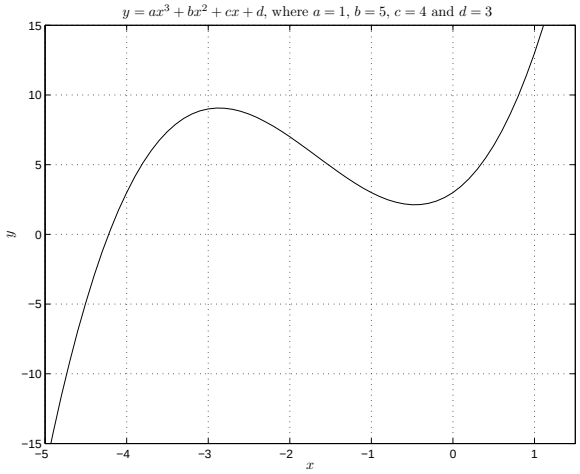


Figure 8 *The cubic function* $y = x^3 + 5x^2 + 4x + 3$.

Example 4. The hyperbolic function of the form $(x - a)(y - b) = c$ has as its asymptotes the lines $x = a$ and $y = b$. Figure 9 shows the graph when $a = 1$, $b = 2$ and $c = 3$. Here the asymptotes are $x = 1$ and $y = 2$.

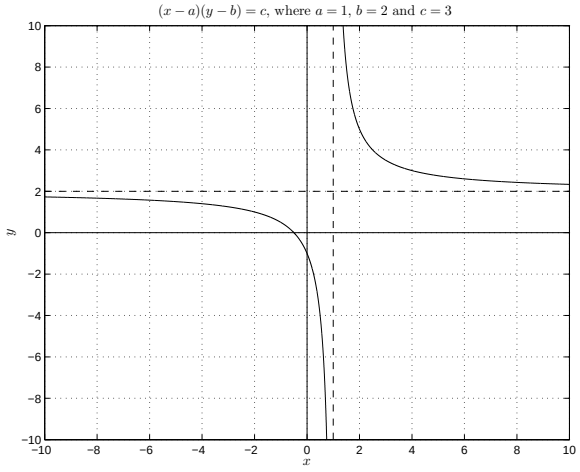


Figure 9 *The hyperbolic function* $(x - 1)(y - 2) = 3$.

Definition 5. The *straight line* has a form $ax + by + c = 0$, or $y = mx + d$ where $m = -a/b$ and $d = -c/b$. The y-intercept is the point where $x = 0$, that is $y = -c/b$ or $y = d$. The x-intercept is the point where $y = 0$, that is $x = -c/a$ or $x = -d/m$.

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Example 5. Given a sum of money, an *isocost line* represents the different combinations of two inputs of production that can be bought. The general formula is $p_k k + p_l l = e$, where k and l are capital and labour, p_k and p_l their respective price, and e the allotted expenditures. In other words, $p_k k + p_l l - e = 0$, or $k = e/p_k - (p_l/p_k)l$. All the values concerned are positive, which means that the graph we are interested in is in the first quadrant only.

Example 6. At the simplest case where there are only two activities to choose from, if x and y are the two activites, then $t_x p_x + t_y p_y = m$, where t_x and t_y the number of hours spent on activities x and y respectively, p_x and p_y are the price per unit hour of x and y, and m is the limit of income.

Definition 6. The *general demand function* is of the form $q_d = \mathrm{f}(p, y, p_s, p_c, t_a, a, \ldots)$, where q_d is the quantity demand of good x, p the price of x, y the income of the consumer, p_s the price of substitute goods, p_c the price of complementary goods, t_a the taste or fashion of the consumer, and a the advertisement level. In its simplest case where all other factors are constant, the *demand equation* takes the form $p = c_1 - c_2 q_d$, where p is the price-, while q_d the quantity demanded of the good x, and c_1 and c_2 are positive constants. The *general supply function* is of the form

$q_s = \mathrm{f}(p, c, p_0, t_e, n, o, \ldots)$, where q_s is the quantity supplied of good x, p the price of x, c the cost of production, p_0 the price of other goods, t_e the available technology, n the number of producers in the market, and o other factors, for example tax and subsidies. The simplified relation for the *supply* is $p = c_1 + c_2 q_s$, where q_s is the quantity of x supplied, and c_1 and c_2 are positive constants.

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Example 7. When a tax of t per unit is imposed, the supply function becomes $p - t = c_1 + c_2 q$, where c_1 and c_2 are positive constants, and the total cost function becomes $c_t = c_f + (k + t)q$, where k is the cost of producing each unit. Here also $c_v = kq$.

Example 8. *Revenue* is the amount of money received when a firm sells its output. The relation is $r_t = pq$, where r_t is the total revenue, p the price and q the quantity.

Definition 7. *Elasticity* ε of x with regard to y means the ratio of the change in x to the change in y. Here y could be some economic variable, for example price or income. Hence, the *price elasticity of demand* is $\varepsilon_d = \frac{\mathrm{d}q_d}{\mathrm{d}p}\frac{p}{q_d}$, the *price elasticity of supply* $\varepsilon_s = \frac{\mathrm{d}q_s}{\mathrm{d}p}\frac{p}{q_s}$. The *point elasticity* is the elasticity value calculated at a point, the *arc elasticity* or the *midpoint elasticity* is the same averaged over an interval. The *arc price elasticity of demand* or *supply* is then $\varepsilon_d = \frac{\mathrm{d}q}{\mathrm{d}p}\frac{p_1+p_2}{q_1+q_2}$. The *arc income elasticity of demand* or *supply* is $\varepsilon_y = \frac{\mathrm{d}q}{\mathrm{d}y}\frac{y_1+y_2}{q_1+q_2}$.

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Elasticity measures the sensitivity or responsiveness of a certain quantity to changes in some variable.

Definition 8. A *function of one independent variable* is a relation in the form $y = \mathrm{f}(x)$ such that there exists one and only one value of y in the range of f for each real number x in the domain of f. The variable y is called the *dependent variable*.

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Definition 9. A function is called *multivalued function* if the opposite to Definition 8 is true, that is there exist more than one values of y for some x.

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Definition 10. An *implicit function* is a function in which both dependent- and independent variables appear on the same side. An *explicit function* is one where the dependent variable is on the left hand side-, and the independent variable on the right hand side of the equation.

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Example 9. Examples of explicit functions are $y = 5x$, $y = x^3 + x^2 - 7$ and $y = e^x + (x+1)\ln x$. Examples of implicit functions are $x + y = 1$, $x^2 + 3xy - y + y^2 = \frac{1}{x+y}$ and $x^2 \ln y = e^y(x + x^3)$.

Definition 11. The *exponential function* has the general form $y = a^x$, where the *base* a is a constant and x is called the *index-*, *power-*, or the *exponent* of the exponential function. The *logarithmic function* is then $\log_a y = x$, providing that both a and y are positive real numbers. We call a^x the x^{th} *power of* a, and call $\log_a y$ the *logarithm* of y to the base a. When the logarithmic base is the natural number, we write $\log_e n = \ln n$.

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The logarithmic function is the inverse of the exponential function and vice versa.

Theorem 1. Let p and q be real numbers, a and b positive numbers, and m and n positive integers. Then,

$$a^p \cdot a^q = a^{p+q}$$

$$\frac{a^p}{a^q} = a^{p-q}$$

$$(a^p)^q = a^{pq}$$

$$a^0 = 1, \text{ provided that } a \neq 0$$

$$a^{-p} = \frac{1}{a^p}$$

$$(ab)^p = a^p b^p$$

$$\sqrt[n]{a} = a^{\frac{1}{n}}$$
$$\sqrt[n]{a^m} = a^{\frac{m}{n}}$$
$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

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Theorem 2.

$$\log_a mn = \log_a m + \log_a n$$
$$\log_a \frac{m}{n} = \log_a m - \log_a n$$
$$\log_a m^p = p \log_a m$$
$$\log_a n = \frac{\log_b n}{\log_b a}$$

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Both Definition 12 and Definition 13 are a definition of limits in calculus. Definition 13 is a more formal and definitive one.

Definition 12. If $\mathrm{f}(x)$ is a function which draws closer to a unique finite real number l for all values of x as the latter draws closer to a, but $x \neq a$, then l is called the *limit* of $\mathrm{f}(x)$ as x approaches a. In notation this is,

$$\lim_{n \to a} \mathrm{f}(x) = l$$

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Definition 13. For a function $\mathrm{f}(x)$, $\lim_{n \to a} \mathrm{f}(x) = l$ if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that $|\mathrm{f}(x) - l| < \epsilon$ whenever $0 < |x - a| < \delta$.

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By Definition 13 we mean that one can get $\mathrm{f}(x)$ to be as close to l as one wish, provided that x gets close enough to a.

Theorem 3. Providing that both $\lim_{x \to a} \mathrm{f}(x)$ and $\lim_{x \to a} \mathrm{g}(x)$ exist, and let k be a constant and n a positive integer, then the *rules of limits* are the following.

$$\lim_{x \to a} k = k$$
$$\lim_{x \to a} x^n = a^n$$
$$\lim_{x \to a} k\mathrm{f}(x) = k \lim_{x \to a} \mathrm{f}(x)$$
$$\lim_{x \to a} [\mathrm{f}(x) \pm \mathrm{g}(x)] = \lim_{x \to a} \mathrm{f}(x) \pm \lim_{x \to a} \mathrm{g}(x)$$
$$\lim_{x \to a} [\mathrm{f}(x) \cdot \mathrm{g}(x)] = \lim_{x \to a} \mathrm{f}(x) \cdot \lim_{x \to a} \mathrm{g}(x)$$
$$\lim_{x \to a} \left[\frac{\mathrm{f}(x)}{\mathrm{g}(x)}\right] = \frac{\lim_{x \to a} \mathrm{f}(x)}{\lim_{x \to a} \mathrm{g}(x)}, \text{ provided that } \lim_{x \to a} \mathrm{g}(x) \neq 0$$
$$\lim_{x \to a} [\mathrm{f}(x)]^n = \left[\lim_{x \to a} \mathrm{f}(x)\right]^n, \text{ provided that } n > 0$$

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Definition 14. Let $y = \mathrm{f}(x)$. Then, the derivative of y with respect to x is,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{h \to 0} \frac{\mathrm{f}(x+h) - \mathrm{f}(x)}{h}$$

The various notations for the derivative include $\mathrm{d}f(x)\mathrm{d}x$, $\frac{\mathrm{d}f}{\mathrm{d}x}$, $f'(x)$, y', $\mathrm{D}y$ and $\mathrm{D}(\mathrm{f}(x))$. The process for obtaining this is called *differentiation*.

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Theorem 4. Let $y = x^n$. Then the *power rule* for differentiation is the following.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = nx^{n-1}$$

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Theorem 5. The derivatives of exponential and logarithmic functions are respectively $\frac{\mathrm{d}e^x}{\mathrm{d}x} = e^x$ and $\frac{\mathrm{d}\ln x}{\mathrm{d}x} = \frac{1}{x}$.

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It turns out that e^x is the only function which is changed by neither differentiation nor integration. Leonhard Euler (1707–83) found that the *natural number* $e = \lim_{n\to\infty}\left(1+\frac{1}{n}^n\right) = 2.71828\ldots$.

Exercise 1. Find the first derivative of the following:

1. $y = 2e^x$	**2.** $y = 2\ln(x)$	**3.** $c_{vt} = \frac{100}{x} + \ln x$
4. $p = 100(1 - e^t)$	**5.** $q = \sqrt[4]{l} + 4$	**6.** $p = 1 + 1\ln q$
7. $q = \frac{\ln p}{8} - \frac{13}{\sqrt{4p}}$	**8.** $c_a = \ln q + \frac{1}{q}$	**9.** $p = 98.5\frac{e^{3t}}{e^t}$
10. $c = x^2 - e^x + \ln x$	**11.** $c = \frac{157}{y} + 0.81e^y$	**12.** $x = 34.3e^t$

Theorem 6. Let y be a function of u, and u a function of x. Then the *chain rule* states that,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}$$

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Exercise 2. Differentiate the following:

1. $p = 5 + \frac{8}{e^{4t}}$	**2.** $y = \frac{5}{2+3e^{4t}}$	**3.** $p = \ln\left(\frac{q^2-2q}{3q}\right)$
4. $y = (3x-5)^7$	**5.** $q = 0.95\sqrt[3]{2}l^2 + 3l$	**6.** $c_t = \sqrt{q^5 + 4q^2}$
7. $y = (2-0.7x)^{-8}$	**8.** $t = 100 + 30e^{0.8y}$	**9.** $y = 11e^{-1.15} + \frac{1}{x+1} - \ln(3x^2 - 2x + 5)$
10. $p = \frac{3}{\sqrt{q}} + 2\ln(q)$	**11.** $s = 123(1 - 3e^{-0.62y})$	**12.** $y = \frac{1}{(5x+6)^{\frac{2}{3}}}$

Theorem 7. If $y = u(x)v(x)$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = v\frac{\mathrm{d}u}{\mathrm{d}x} + u\frac{\mathrm{d}v}{\mathrm{d}x}$$

Similarly, if $y = u(x)v(x)w(x)$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = uv\frac{\mathrm{d}w}{\mathrm{d}x} + uw\frac{\mathrm{d}v}{\mathrm{d}x} + vw\frac{\mathrm{d}u}{\mathrm{d}x}$$

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Exercise 3. Find the first derivatives of the following:

1. $c = 3q\sqrt{2+q}$	**2.** $p = \frac{2}{33}x^2e^{x-7}$	**3.** $y = x^{1.3}\sqrt{4x}$
4. $y = (t^3)e^{2t} + 5$	**5.** $s = 10ye^{-0.1y}$	**6.** $c_a = \frac{1}{q} - q\ln q + \sqrt{q}$
7. $x = y^3\ln(y^3) + \sqrt{3y}\ln y$	**8.** $I = \frac{1}{y}\ln(3y+2)$	**9.** $p = q^2e^{q-3}\ln(q^2-8)$
10. $c_t = \frac{1}{q^3} + q\sqrt{q-1} - q + 100$	**11.** $y = t^2e^t + 10\ln t$	**12.** $s = \frac{e^y\ln y}{y}$

Theorem 8. If $y = \frac{u(x)}{v(x)}$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$$

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Exercise 4. Find the first derivative of the following and simplify your answer:

1. $y = \frac{t-1}{t+1}$ **2.** $t = \frac{1-t}{1+t}$ **3.** $t = \frac{\ln y}{y}$

4. $y = \frac{x \ln x + 1}{x}$ **5.** $p = 100 - \frac{q}{q+1}$ **6.** $c = 500\left(1 - \frac{e^{-1.6y}}{y}\right)$

7. $p = \frac{q^3}{q+1}$ **8.** $c_t = \frac{q^2 e^{6-q}}{(x^3+x^2+x+1)\ln q}$ **9.** $y = \frac{\sqrt{t-1}}{t}$

Definition 15. The *total cost* c_t comprises a fixed cost and a variable cost, that is $c_t = c_f + c_v$. The *total revenue* r_t is price times output, $r_t = pq$.

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Definition 16. The *marginal cost* c_m is the change in total cost caused by the production of an additional unit. The *marginal revenue* r_m is the change in total revenue coming from the sale of an extra good. In other words, $c_m = \frac{\mathrm{d}c_t}{\mathrm{d}q}$ and $r_m = \frac{\mathrm{d}r_t}{\mathrm{d}q}$, where q is the output.

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Definition 17. The *average cost* c_a is the total cost per unit output, $c_a = \frac{c_t}{q}$. The *average revenue* r_a is the total revenue per unit output, $r_a = \frac{r_t}{q}$.

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Example 10. From Definition 15, $r_t = pq$, and from Definition 17, $r_t = r_a q$, therefore $p - r_a$. From Definition 17, $c_a = \frac{c_t}{q}$, and from Definition 15, $c_a = c_f + c_v$. Therefore $c_a = c_{af} + c_{av}$, where the average fixed cost $c_{af} = \frac{c_f}{q}$ and the average variable cost $c_{av} = \frac{c_v}{q}$.

Procedure 1 calculates and draws graphs of the total-, average-, and marginal costs, which are respectively c_t, c_a and c_m. Similar graphs of other quantities may be drawn likewise, for example ones for the total-, average-, and marginal products.

Procedure 1 *Total-, average-, and marginal graphs*

Given: $c_t(x)$
$c_a \leftarrow \frac{c_t}{x}$
$c_m \leftarrow c_t'$
for each $f \in \{c_t, c_a, c_m\}$ **do**
  find the critical values $\mathbf{x}_c$ for $f' = 0$
  find f''
  $n \leftarrow |\mathbf{x}_c|$
  for $i = 1$ to n **do**
    if $f''(x_i^c) > 0$ **then**
      $f(x_i^c)$ is convex and is the relative minimum of f
    elseif $f''(x_i^c) < 0$ **then**
      $f(x_i^c)$ is concave and is the relative maximum of f
    endif
  endfor
  find inflection points $\mathbf{x}_f$ from $f'' = 0$
endfor
plot $c_t(x)$, then $c_a(x)$ and $c_m(x)$

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Problem 1. Let the total cost function be $c = q^3 - 40q^2 + 800$. Sketch the graph to show the relationship between total-, average-, and marginal costs. If $c_t = \mathrm{f}(q)$, then $c_m = \mathrm{d}c_t/\mathrm{d}q$, and if $r_t = \mathrm{f}(q)$, then $r_m = \mathrm{d}r_t/\mathrm{d}q$, where c_t is the total cost, r_t the total revenue, and q the level of output.

Problem 2. The demand function for a monopolist is $q = 100 - 4p$. Find r_t, r_m and r_a, then evaluate these at $q = 10$ and explain the results. Further, find q when $r_a = 0$, then find r_m when $r_a = 0$, and then explain whether one should sell at this value of q. Draw graphs of r_t, r_m and r_a on the same diagram.

Problem 3. The demand function for a good is $p = 200 - q^{1.8}$. Find r_t, r_m and r_a, and then compare the slopes of the r_m- and r_a curves. Evaluate r_t, r_m and r_a at $q = 11$ and at $q = 26$, and give an explanation of the values obtained. Find q when $r_m = 0$, and find q when $r_a = 0$. When

does the sales of more units begin to reduce the total revenue? Draw graphs of r_t, r_m and r_a on the same diagram.

Problem 4. The fixed costs of a firm are 2000 and the variable costs are $4.7q$. Write the equation for c_t and evaluate the c_t at $q = 35$. Write the equation for c_m and evaluate the c_m at $q = 35$. Explain the meaning of c_m for this function.

Problem 5. The average cost function of a firm is given by $c_a = q^2 - 10q + \frac{210}{q} + 66$. Write expressions for c_t and c_m, and then calculate c_t at $q = 17$. Write the equations for c_f and c_{vt}.

Problem 6. The average cost function of a firm is $c_a = 97 + \frac{21}{q}$. Differentiate c_a with respect to q, then describe and explain how the former changes with the change in the latter.

Definition 18. The relationship between input and output is called a *production function*, $q = \mathrm{f}(l, k, r, t_e, s, e, \ldots)$, where l is labour, k phical capital such as buildings and machines, r raw materials, t_e technology, s land, and e enterprise. Assuming a short period of time, then l becomes the only independent variable and the other remaining factors are parameters, that is fixed, and therefore $q = \mathrm{f}(l)$. Then the *marginal product of labour* is $p_{lm} = \frac{\mathrm{d}q}{\mathrm{d}l}$, and the *average product of labour* is $p_{la} = \frac{q}{l}$.

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The value p_{la} in Definition 18 is a measure of productivity.

Definition 19. The *marginal propensity to consume* is $p_{cm} = \frac{\mathrm{d}c}{\mathrm{d}y}$. The *marginal propensity to save* is $p_{sm} = \frac{\mathrm{d}s}{\mathrm{d}y}$. The *average propensity to consume* is $p_{ca} = \frac{c}{y}$. The *average propensity to save* is $p_{sa} = \frac{s}{y}$. Here y is the income, c the consumption, and s the saving.

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Example 11. Since $y = c + s$, it follows that $p_{ca} + p_{sa} = 1$. Further, $\frac{\mathrm{d}y}{\mathrm{d}y} = \frac{\mathrm{d}c}{\mathrm{d}y} + \frac{\mathrm{d}s}{\mathrm{d}y}$, and hence $1 = p_{cm} + p_{sm}$.

Example 12. The consumption function may be described by $c = c_0 + by$, where c_0 and b are positive constants. Then $p_{ca} > p_{cm}$ and $p_{sm} > p_{sa}$.

Definition 20. The *profit* is $\pi = r_t - c_t$. At the *break-even point* $\pi = 0$, that is $r_t = c_t$.

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Problem 7. Find the elasticity of demand when $q = \frac{a}{p^c}$, where a and c are constants.

Problem 8. The demand function for train journeys is $q = \frac{1600}{p^{1.3}}$, where q is the number of passengers in thousands. Find the elasticity of demand in this case. Determine the percentage change in demand when the fare increases or decreases by 10 per cent.

Problem 9. The demand for mineral water is given by $q = 100 - 2p$, where p is the price per bottle and q the number of bottles demanded. Find the expressions for r_t, r_m and r_a. Find the quantity and price that maximise the revenue. Find an equation for the price elasticity of demand firstly in terms of p, and then again in terms of q. Show that r_t is maximum and r_m is zero when $\varepsilon_d = -1$, and find q at this ε_d.

Problem 10. A demand for a wine is given by $p = 1200e^{-0.03q}$, where p is the price per bottle and q the number of bottles demanded. Write the equations for r_t, r_m and r_a. Find the price and quantity when the revenue is maximised. Show that $\varepsilon_d = -1$ when r_t is maximised.

Problem 11. Derive the equation

$$r_m = p\left(1 + \frac{1}{\varepsilon_d}\right)$$

Problem 12. Show that the profit becomes maximised when

$$p = \frac{c_m}{\left(1 + \frac{1}{\varepsilon_d}\right)}$$

Problem 13. A shop selling shirts has a demand function $p = 300 - 10q$ and a total cost function $c_t = 150 + 8q$. Find the equations for r_t and π. Find the number of shirts which must be sold

firstly in order to maximise the profit, and then again to maximise the total revenue. Show that $r_m = r_c$ when the profit is maximised.

Problem 14. The average cost function of a mobile phone is $c_a = 10 + \frac{5000}{q}$, and the average revenue function for the same is $r_a = 35$. Find r_t, c_t, r_m and c_m. How many mobile phones must be made and sold in order to break-even? Find the profit function, and show that neither profit nor revenue has a maximum. Explain using r_m and c_m why there is no maximum. Plot the graphs for r_t, c_t and π on one diagram, and plot r_m and c_m on another diagram. Comment on the graphs.

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